

# IDEMPOTENTS WITH SMALL NORMS

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ABSTRACT. Let  $\Gamma$  be a locally compact group. We answer two questions left open in [7] and [9]:

- (i) For abelian  $\Gamma$ , we prove that if  $\chi_S \in B(\Gamma)$  is an idempotent with norm  $\|\chi_S\| < \frac{4}{3}$ , then  $S$  is the union of two cosets of an open subgroup of  $\Gamma$ .
- (ii) For general  $\Gamma$ , we prove that if  $\chi_S \in M_{cb}A(\Gamma)$  is an idempotent with norm  $\|\chi_S\|_{cb} < \frac{1+\sqrt{2}}{2}$ , then  $S$  is an open coset in  $\Gamma$ .

## 1. INTRODUCTION

In his 1968 papers, Saeki determined idempotent measures on a locally compact abelian group  $G$  with small norms. These are equivalent to determining idempotent functions in the Fourier–Stieltjes algebras  $B(\Gamma)$  on a locally compact abelian group  $\Gamma$  with small norms (where  $\Gamma$  and  $G$  could be taken as Pontryagin duals of each other). The statements of Saeki’s results in the Fourier–Stieltjes setting are:

**Theorem 1.1** (Saeki). *Let  $\Gamma$  be a locally compact abelian group, and let  $\varphi$  be an idempotent function in  $B(\Gamma)$  so that  $\varphi = \chi_S$  for some nonempty  $S \subseteq \Gamma$ . Then*

- (i) ([6]) *If  $\|\varphi\| < \frac{1+\sqrt{2}}{2}$ , then  $S$  is an open coset of  $\Gamma$ .*
- (ii) ([7]) *If  $\|\varphi\| \in (1, \frac{\sqrt{17}+1}{4})$ , then  $S$  is the union of two cosets of an open subgroup of  $\Gamma$  but is not a coset itself.*

For abelian  $\Gamma$ , it is well-known (see [5, page 73]) that if  $S$  is an open coset of  $\Gamma$ , then  $\|\chi_S\| = 1$ , and whereas if  $S$  is the union of two cosets of an open subgroup of  $\Gamma$  but is not a coset itself, then

$$(1.1) \quad \|\chi_S\| = \begin{cases} \frac{2}{q \sin(\pi/2q)} & \text{if } q \text{ is odd} \\ \frac{2}{q \tan(\pi/2q)} & \text{if } q \text{ is even} \\ \frac{4}{\pi} & \text{if } q = \infty \end{cases}$$

where  $q$  is the “relative order” of the two cosets forming  $S$ . The largest value in (1.1) is  $4/3$  when  $q = 3$  and the smallest one is  $\frac{1+\sqrt{2}}{2}$  when  $q = 4$ . In particular, the number  $\frac{1+\sqrt{2}}{2}$  in Theorem 1.1 (i) is sharp.

The paper [7] asked whether or not the interval  $(1, \frac{\sqrt{17}+1}{4})$  in Theorem 1.1 (ii) could be increased to  $(1, \frac{4}{3})$ , and we answer this in affirmative in Theorem 3.4. Note that the interval  $(1, \frac{4}{3})$  is sharp because of the discussion in the previous paragraph

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and also since there are idempotents  $\chi_S$  of  $B(\Gamma)$  with  $\|\chi_S\| = 4/3$  but  $S$  is not any union of two cosets of open subgroup of  $\Gamma$  (see the last paragraph of [7]).

Lesser is known about idempotents in  $B(\Gamma)$  with small norms for general locally compact group  $\Gamma$ . Ilie and Spronk [3] proved that  $\chi_S$  is an idempotent in  $B(\Gamma)$  with  $\|\chi_S\| = 1$  if and only if  $S$  is an open coset of  $\Gamma$ . More generally, Stan proved the following.

**Theorem 1.2** (Stan [9]). *Let  $\Gamma$  be a locally compact group, and let  $\varphi$  be an idempotent function in  $M_{cb}A(\Gamma)$  so that  $\varphi = \chi_S$  for some nonempty  $S \subseteq \Gamma$ . If  $\|\varphi\|_{cb} < \frac{2}{\sqrt{3}}$ , then  $S$  is an open coset of  $\Gamma$ , and in which case  $\|\varphi\|_{cb} = 1$ .*

Here  $M_{cb}A(\Gamma)$  is the completely bounded multiplier algebra  $M_{cb}A(\Gamma)$  of the Fourier algebra  $A(\Gamma)$  and is defined as follows. Since the Fourier algebra  $A(\Gamma)$  is the predual of the group von Neumann algebra  $VN(\Gamma)$ , it has the canonical operator space structure, which makes it a completely contractive operator algebra (see the monograph [2] for more details). The completely bounded multiplier algebra  $M_{cb}A(\Gamma)$  of  $A(\Gamma)$  consists of those continuous functions  $\varphi : \Gamma \rightarrow \mathbb{C}$  such that the mapping

$$f \mapsto \varphi \cdot f, A(\Gamma) \rightarrow A(\Gamma),$$

is completely bounded, and its completely bounded norm is denoted as  $\|\varphi\|_{cb}$  (whereas the Fourier–Stieltjes norm on  $\Gamma$  will be simply denoted as  $\|\cdot\|$  in this paper). In general, we have

$$B(\Gamma) \subseteq M_{cb}A(\Gamma), \quad \text{with a decreasing of norms,}$$

but, for amenable locally compact groups  $\Gamma$ ,

$$B(\Gamma) = M_{cb}A(\Gamma), \quad \text{isometrically.}$$

Thus an idempotent of  $B(\Gamma)$  with a small norm is always an idempotent of  $M_{cb}A(\Gamma)$  with a small(er) norm.

In Theorem 2.2, we increase the number  $\frac{2}{\sqrt{3}}$  in Stan’s Theorem 1.2 to the sharp bound of  $\frac{1+\sqrt{2}}{2}$ , and so obtaining a generalisation of the first mentioned result of Saeki, Theorem 1.1 (i), to general locally compact groups.

## 2. IDEMPOTENTS OF $M_{cb}A(\Gamma)$ WITH NORM LESSER THAN $\frac{1+\sqrt{2}}{2}$

In this section, let  $\Gamma$  be any locally compact group, and let  $\chi_S$  be an idempotent of  $M_{cb}A(\Gamma)$  with  $\|\chi_S\|_{cb} < \frac{1+\sqrt{2}}{2}$ . Our aim is to show that  $S$  is an open coset of  $\Gamma$ . It is obvious that  $S$  is open, and so, it remains to show that  $S$  is a coset of  $\Gamma$ . By [8, Corollary 6.3 (i)], it is sufficient for us to consider the case where  $\Gamma$  is discrete.

We first make a simple observation.

**Lemma 2.1.** *For any  $s \in S$  and  $t \in \Gamma$ , if  $st \in S$  (resp.  $ts \in S$ ), then  $st^n \in S$  for every  $n \in \mathbb{Z}$  (resp.  $t^n s \in S$  for every  $n \in \mathbb{Z}$ ).*

*Proof.* By translation, we may (and shall) suppose that  $s = e$ , the identity of  $\Gamma$ . Consider  $\Gamma_0$  be the (abelian) group generated by  $t$ , then

$$\|\chi_{S \cap \Gamma_0}\| = \|\chi_{S \cap \Gamma_0}\|_{cb} \leq \|\chi_S\|_{cb} < \frac{1+\sqrt{2}}{2}.$$

So by Saeki’s Theorem 1.1 (i), we see that  $S \cap \Gamma_0 = \Gamma_0$ . This gives the lemma.  $\square$

To get more information out of the assumption on  $\|\chi_S\|_{cb}$ , we shall follow in the footsteps of [9] and use the connection shown in [1] between the norm  $\|\cdot\|_{cb}$  of  $M_{cb}A(\Gamma)$  and the Schur multiplier norms described below.

Denote by  $\mathcal{K}_0$  the space of matrices that have only finitely many nonzero entries whose rows and columns are indexed by  $\Gamma$ . Then  $\mathcal{K}_0$  is identified with a subspace of  $\mathcal{B}(\ell^2(\Gamma))$ . Recall that the *Schur multiplication* of two matrices  $A$  and  $X$ , indexed by  $\Gamma$ , is defined as

$$(A \bullet X)(s, t) := A(s, t)X(s, t) \quad (s, t \in \Gamma),$$

and for each matrix  $A$ , indexed by  $\Gamma$ , its *Schur multiplier norm* is

$$\|A\|_{\text{Schur}} := \sup \left\{ \frac{\|A \bullet X\|_{\mathcal{B}(\ell^2(\Gamma))}}{\|X\|_{\mathcal{B}(\ell^2(\Gamma))}} : X \in \mathcal{K}_0 \right\}.$$

Of course, this discussion works for any index set  $\Gamma$ , and a particular matrix that is useful for us is the following  $3 \times 3$  matrix

$$(2.1) \quad F_0 := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Using the orthogonal matrix  $U := \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 1 & -1 \\ \sqrt{2} & -1 & 1 \end{pmatrix}$  and the vector  $\xi := \frac{1}{2} \begin{pmatrix} \sqrt{2} \\ 1 \\ 1 \end{pmatrix}$ ,

we see that

$$\|F_0\|_{\text{Schur}} \geq \|A \bullet U\|_{\mathcal{B}(\ell^2)} \geq \frac{\|(A \bullet U)\xi\|_{\ell^2}}{\|\xi\|_{\ell^2}} = \frac{\sqrt{26}}{4} > \frac{1 + \sqrt{2}}{2}.$$

As a matter of fact, it is proved in [4, Proposition 5.1(8)] that  $\|F_0\|_{\text{Schur}} = \frac{9}{7}$ , but the above simple calculation is sufficient for our purpose. Hence, any matrix  $A$  that has a submatrix of the form  $F_0$  in (2.1) must satisfy

$$\|A\|_{\text{Schur}} > \frac{1 + \sqrt{2}}{2}.$$

Returning to our problem on the group  $\Gamma$ , each function  $\varphi : \Gamma \rightarrow \mathbb{C}$  defines a matrix  $M_\varphi$ , indexed by  $\Gamma$ , by setting

$$M_\varphi(s, t) := \varphi(s^{-1}t) \quad (s, t \in \Gamma).$$

An important fact shown in [1] is that

$$(2.2) \quad \|\varphi\|_{cb} = \|M_\varphi\|_{\text{Schur}}.$$

Hence, the previous paragraph implies that  $M_{\chi_S}$  cannot have (2.1) as a submatrix.

Our main result of this section is the following.

**Theorem 2.2.** *Let  $\Gamma$  be a locally compact group, and let  $\varphi$  be an idempotent function in  $M_{cb}A(\Gamma)$  so that  $\varphi = \chi_S$  for some nonempty  $S \subseteq \Gamma$ . If  $\|\varphi\|_{cb} < \frac{1+\sqrt{2}}{2}$ , then  $S$  is an open coset of  $\Gamma$ .*

*Proof.* As discussed above, we may (and shall) suppose that  $\Gamma$  is discrete. Also, applying a translation if necessary, we suppose that  $e \in S$ . So it remains to prove that  $S$  is a subgroup of  $\Gamma$ .

By Lemma 2.1, we see that if  $u \in S$ , then  $u^n \in S$  for every  $n \in \mathbb{Z}$ . Thus it remains to show that  $S$  is closed under multiplication.

We next *claim* that if  $u, v \in S$ , then either  $uv \in S$  or  $vu \in S$ . Indeed, assume towards a contradiction that both  $uv \notin S$  and  $vu \notin S$ . Then the submatrix of  $M_{\chi_S}$  with rows  $e, u^{-1}, v^{-1}$  and columns  $e, u, v$  is

$$\begin{pmatrix} \chi_S(e) & \chi_S(u) & \chi_S(v) \\ \chi_S(s) & \chi_S(u^2) & \chi_S(uv) \\ \chi_S(v) & \chi_S(vu) & \chi_S(v^2) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

by the previous paragraph. This contradicts the previous discussion.

Finally, suppose that  $u, v \in S$ , the proof is completed if we can show that  $uv \in S$ . The claim shows that either  $uv \in S$  or  $vu \in S$ . Assume the latter holds, then from Lemma 2.1 with  $s = v$  and  $t = u$ , we obtain that  $vu^{-1} \in S$ . Since we must have  $u^{-1} \in S$ , this in turn implies, by a similar argument, that  $v^{-1}u^{-1} \in S$ . But then, since  $v^{-1}u^{-1} = (uv)^{-1}$ , we must have  $uv \in S$ . Hence, in any case,  $uv \in S$ , and the proof is completed.  $\square$

### 3. IDEMPOTENTS OF $B(\Gamma)$ WITH NORM LESSER THAN $\frac{4}{3}$ , FOR ABELIAN $\Gamma$

In this section, let  $\Gamma = (\Gamma, 0, +)$  be a locally compact abelian group. We aim to strengthen Saeki's Theorem 1.1 (ii) by enlarging his range of  $(1, \frac{\sqrt{17}+1}{4})$  to the optimal  $(1, \frac{4}{3})$ . So let  $\chi_S$  be an idempotent function in  $B(\Gamma)$  with  $\|\chi_S\| \leq \frac{4}{3}$ .

Actually, in [7], Saeki works with idempotents of the measure algebra  $M(G)$  on a locally compact abelian group  $G$ , and so, our  $\Gamma$  and his  $G$  could be considered as the Pontryagin's duals of each other. Thus  $B(\Gamma) \cong M(G)$  isometrically, and we denote by  $\mu$  the idempotent measure in  $M(G)$  that corresponds to  $\chi_S$ .

As in the previous section, we may reduce our problem to the case where  $\Gamma$  is discrete. Thus suppose that  $\Gamma$  is discrete, and so  $G$  is compact.

Saeki's proof of Theorem 1.1 (ii) in [7] invokes the following lemma several times.

**Lemma 3.1** (Saeki). *Assume as above. Suppose there exists  $u$  and  $v$  in  $S$  and  $w$  in  $\Gamma$  such that  $u + w$  belongs to  $S$  but neither  $v + w$  nor  $v - w$  belongs to  $S$ . Then we have  $\|\mu\| \geq \frac{\sqrt{17}+1}{4}$ .*

In the main argument, Saeki uses this lemma to show that if  $S$  is not the union of two cosets of a subgroup in  $\Gamma$ , then  $\|\mu\| \geq \frac{\sqrt{17}+1}{4}$ . The argument used breaks the problem up into many cases, and in the cases where this lemma is not used, it is always shown that in fact  $\|\mu\| \geq \frac{4}{3}$ . Thus if we can strengthen this lemma, then Theorem 1.1 (ii) is strengthened also. In fact, we prove the following.

**Lemma 3.2.** *Assume as above. Suppose there exists  $u, v \in S$ , and  $w \in \Gamma$ , such that  $u + w \in S$ , and  $v + w, v - w \notin S$ . Then  $\|\chi_S\| = \|\mu\| \geq \frac{4}{3}$ .*

*Proof.* Let us define a function  $f \in C(G)$  to be

$$f(x) = (x, u)[2 + 2(x, w) + \frac{1}{2}(x, -w)] + (x, v)[2 - (x, w) - (x, -w)].$$

If  $u - w \in S$ , we get  $|\int f(x) d\mu(x)| = \frac{13}{2}$ , otherwise it will simply be 6. Next, we calculate the uniform norm  $\|f\|_G$  of  $f$ , by taking  $x \in G$ , and set  $(x, w) =: e^{i\theta}$ . Then we see

$$\begin{aligned} |f(x)| &\leq \left| 2 + 2e^{i\theta} + \frac{1}{2}e^{-i\theta} \right| + |2 - e^{i\theta} - e^{-i\theta}| \\ &= \sqrt{\frac{25}{4} + 10\cos(\theta) + 4\cos^2(\theta) + 2 - 2\cos(\theta)} = \frac{9}{2}. \end{aligned}$$

Thus  $|f|_G \leq \frac{9}{2}$ . Hence

$$\|\mu\| \geq \frac{|\int_G f(x) d\mu(x)|}{|f|_G} \geq \frac{6}{9/2} = \frac{4}{3}.$$

This proves our lemma.  $\square$

Now we can get our desired result.

**Theorem 3.3.** *Let  $\Gamma$  be a locally compact abelian group, and let  $\varphi$  be an idempotent function in  $B(\Gamma)$  so that  $\varphi = \chi_S$  for some nonempty  $S \subseteq \Gamma$ . If  $\|\varphi\| \in (1, \frac{4}{3})$ , then  $S$  is the union of two cosets of some open subgroup of  $\Gamma$  but is not a coset itself.  $\square$*

*Proof.* This follows from the previous discussion with the argument of [7].  $\square$

Equivalently, the above translates into the following:

**Theorem 3.4.** *Let  $G$  be a locally compact abelian group, and let  $\mu$  be an idempotent measure on  $G$  with  $\|\mu\| \in (1, \frac{4}{3})$ . Then*

$$d\mu(x) = [(-x, \gamma_1) + (-x, \gamma_2)] dm(x)$$

*where  $m$  is the Haar measure of some compact subgroup  $H$  of  $G$ , and  $\gamma_1, \gamma_2$  are distinct characters of  $H$ .  $\square$*

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